EXPONENTIALLY TWISTED DE RHAM COHOMOLOGY AND RIGID COHOMOLOGY

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ABSTRACT. A comparison theorem between exponentially twisted de Rham cohomology and rigid cohomology with coefficients in a Dwork crystal is proved.

INTRODUCTION

0.1. Cohomology groups with exponential twists. Let k be a field. Let $f\colon X \to \mathbf{A}^1_k$ be a morphism of algebraic varieties over k. Depending upon what k is, one can consider the following realizations of the "exponential motive" (in the sense of Fresán–Jossen [20]) associated with the function f.

(1) Betti realization. When k is the field $\mathbf{C}$ of complex numbers, one can consider the relative singular cohomology $\mathrm{H}^\bullet(X^{\mathrm{an}}, f^{-1}(t)^{\mathrm{an}})$ (here $t \in \mathbf{C}$ and $|t|$ is sufficiently large, in fact any *typical value*[1] of f will do). The Betti realization has an integral structure.

(2) De Rham realization. For an arbitrary k, one can consider the exponentially twisted de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{DR}}(X, \nabla_f)$, where ∇_f is the integrable connection on the trivial module $\mathcal{O}_X$ defined by $\nabla_f(h) = \mathrm{d}h + h\mathrm{d}f$. It should be brought to the reader's attention that the connection ∇_f has irregular singularity, thus does not fit into the picture of [11].

When $k = \mathbf{C}$, and when the Betti cohomology is taken $\mathbf{C}$ as its coefficient ring, it is known that the cohomology groups in (1) and (2) are isomorphic. This theorem could be attributed to Deligne–Malgrange [13, pp. 79, 81, 87], Dimca–Saito (the upshot is [15, Proposition 2.8]), and Sabbah [30].

(3) Rigid analytic de Rham realization. When k is a field equipped with a complete ultrametric, one can consider the rigid analytic version of the twisted de Rham cohomology $\mathrm{H}^\bullet_{\mathrm{DR}}(X^{\mathrm{an}}, \nabla_f)$.

When k is of characteristic 0, it follows from the André–Baldassarri comparison theorem [2, Theorem 6.1] that (2) and (3) are isomorphic[2]. Note that the complex analytic version of this result is false even in the simplest situation $X = \mathbf{A}^1$, $f = \mathrm{Id}$, since ∇_f has irregular singularity at infinity (indeed, the complex analytification of ∇_f is isomorphic to the trivial connection: $\nabla_f^{\mathrm{an}} = \mathrm{e}^{-f} \circ \mathrm{d} \circ \mathrm{e}^f$).

(4) ℓ-adic realization. Assume that k is a finite field of characteristic $p > 0$. Fix a nontrivial additive character $\psi\colon k \to \mathbf{C}^*$, and an algebraic closure k^{a} of k. Let

[1] We say t is a typical value of f if t falls in the largest open subset of $\mathbf{A}^{1,\mathrm{an}}$ on which $R^i f_* \mathbf{Q}$ are locally constant for all i.

[2] The André–Baldassarri theorem as stated in [2] does not immediately imply the said isomorphy, as the variety we consider is not assumed to be defined over a number field. Instead of walking through their dévissage argument, we shall present an alternative proof of it.

k_m be the subfield of k^{a} such that $[k_m : k] = m$. One can consider the L-series associated with the exponential sums defined by f:

$$S_m(f) = \sum_{x \in X(k_m)} \psi(\mathrm{Tr}_{k_m/k} f(x)); \quad L_f(t) = \exp\left\{\sum_{m=1}^{\infty} \frac{S_m}{m} t^m\right\}.$$

By a theorem of Grothendieck, this L-series is the (super) determinant of the Frobenius operation on a twisted $\overline{\mathbf{Q}}_\ell$-étale cohomology theory.

(5) Crystalline realization. When k is a perfect field of characteristic $p > 0$, one can consider the rigid cohomology $\mathrm{H}^\bullet_{\mathrm{rig}}(X/K, f^*\mathcal{L}_\pi)$, or rigid cohomology with compact support $\mathrm{H}^\bullet_{\mathrm{rig,c}}(X/K, f^*\mathcal{L}_\pi^\vee)$. Here, $\mathcal{L}_\pi$ is a certain overconvergent isocrystal on $\mathbf{A}^1_k$ called "Dwork isocrystal", and $\mathcal{L}_\pi^\vee$ its dual isocrystal. The two cohomology groups are related by Poincaré duality for rigid cohomology with twisted coefficients.

By a theorem of Etesse and Le Stum [19] (see also [4]), the compactly supported rigid cohomology admits a Frobenius operation which, when k is finite, could determine the L-series as in Item (4).

In these notes, we shall prove a comparison theorem between (0.1/2) and (0.1/5), thus building a bridge between topology and arithmetic.

To state the result, let us set up some notation.

- Let X be a smooth scheme over a finitely generated $\mathbf{Z}$-algebra R of characteristic 0 which is an integral domain. Let $f\colon X \to \mathbf{A}^1_R$ be a morphism. Let $\sigma\colon R \to \mathbf{C}$ be any embedding of R into the field $\mathbf{C}$ of complex numbers.
- For each maximal $\mathfrak{p}$ of R, let $\kappa(\mathfrak{p})$ be the residue field of $\mathfrak{p}$, let $K_\mathfrak{p}$ be the field of fractions of $W(\kappa(\mathfrak{p}))[\zeta_p]$, the ring of Witt vectors of $\kappa(\mathfrak{p})$ with p^{th} roots of unity adjoined.
- For an R-algebra R', we still use f to denote the morphism $X_{R'} = X \times_R \mathrm{Spec}(R') \to \mathbf{A}^1_{R'}$.

The most accessible statement of our result is the following.

Theorem 0.2. *There is a dense Zariski open subset U of* $\mathrm{Spec}(R)$ *such that for any closed point $\mathfrak{p} \in U$, any integer m, the $K_\mathfrak{p}$-dimension of the rigid cohomology $\mathrm{H}^m_{\mathrm{rig}}(X_{\kappa(\mathfrak{p})}/K_\mathfrak{p}, f^*\mathcal{L}_\pi)$ equals the complex dimension of the complex vector space $\mathrm{H}^m_{\mathrm{DR}}(X \times_{R,\sigma} \mathrm{Spec}(\mathbf{C}), \nabla_f)$.*

In the main text, we shall give a precise condition on which $\mathfrak{p}$ is good for the comparison theorem to hold based on ramifications of f at infinity. See Theorem 3.15.

We shall also prove a version of Theorem 0.2 comparing the algebraic Higgs cohomology associated with f and an overconvergent Higgs cohomology. See Proposition 5.4. This latter comparison theorem has significant weaker restrictions on the shape of f.

When X is an open subspace of $\mathbf{P}^1$, the theorem is due to Phillepe Robba [29]. When X is a curve, the theorem is a simple corollary of Joe Kramer-Miller's theorem [24, Theorem 1.1], see Example 4.3.

Theorem 0.2 (or rather its stronger version, Theorem 3.15) is desirable, because it seems that in the literature, the methods used to study exponential sums are either toric in nature, or only applicable to "tame" functions (e.g., Newton nondegenerate Laurent polynomials, or Newton nondegenerate and convenient polynomials), whereas Theorem 3.15 is unconditional. In practice, Theorem 3.15 allows us to calculate the dimension of the rigid cohomology, hence the degree of the L-series, using topological methods. See §4 for some concrete examples. Here we only explain one general procedure for producing examples on which Theorem 0.2 is applicable.

Example 0.3 ("Standard situation"). Let P be a smooth projective variety of pure dimension n over a number field $\mathbf{K}$. Let $\mathcal{L}_1, \dots, \mathcal{L}_r$ be invertible sheaves on P. Suppose we are given sections $s_i \in \mathrm{H}^0(X, \mathcal{L}_i)$ of these invertible sheaves such that the zero loci $D_i = \{s_i = 0\}$ form a divisor with strict normal crossings.

Let $X = P - \bigcup_{i=1}^r D_i$. Let $s_0 \in \mathrm{H}^0(X, \mathcal{L}_1^{e_1} \otimes \cdots \otimes \mathcal{L}_r^{e_r})$, and $s_\infty = s_1^{e_1} \cdots s_r^{e_r}$. Assume that $X_0 = \{s_0 = 0\}$ is a smooth subvariety of P. Write $X_\infty = \{s_\infty = 0\}$ be the vanishing divisor of s_∞.

Then $g(x) = s_0(x)/s_\infty(x)$ is a well-defined regular function on X. We assume in addition the divisor $X_0 = \{s_0 = 0\}$ is transverse to all the intersections $D_{i_1} \cap \cdots \cap D_{i_m}$.

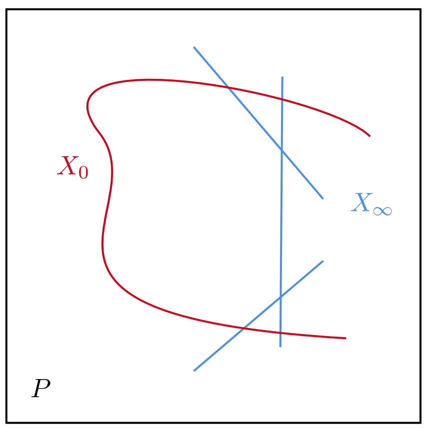

FIGURE 1. Standard situation

If $\mathfrak{p}$ is a prime of $\mathcal{O}_{\mathbf{K}}$ such that

1. the logarithmic pairs (P, X_∞) and $(X_0, X_0 \cap X_\infty)$ all have good reductions at $\mathfrak{p}$, and
2. the residue characteristic of $\mathfrak{p}$ does not divide $e_1 \cdots e_r$,

then Theorem 0.2 (for the function $g\colon X \to \mathbf{A}^1$) is valid at $\mathfrak{p}$. Moreover, if $\bigotimes_{i=1}^r \mathcal{L}^{\otimes e_i}$ is ample, then the rigid cohomology is nonzero in degree n only.

For more details, we refer the reader to Corollary 4.11.

0.4. Previously known theorems about degrees of L-series.

(a) *ℓ-adic theorems.* When k is a finite field, and when $X = \mathbf{G}_{\mathrm{m}}^n$, Denef and Loeser studied the étale cohomology appeared in (0.1/4). They showed that [14, Theorem 1.3] if f is "nondegenerate with respect to its Newton polyhedron at infinity", then the twisted étale cohomology is acyclic except in degree n, and the Frobenius eigenvalues are pure of weight n. In general, they were able to show that the Euler characteristic of the étale cohomology agrees with the Euler characteristic of the algebraic de Rham cohomology (0.1/2) defined by a Teichmüller lift of f (the combinatorial formulas for both theories match).

In the standard situation (0.3), assuming the invertible sheaves $\mathcal{L}_i$ are ample, the étale cohomology associated with the exponential sums of the function g was studied by Katz, see [21, Theorem 5.4.1]. In this case, he proved the L-series is a polynomial or a reciprocal of a polynomial, whose degree can be calculated using Chern classes. We could also deduce these results from Theorem 0.2.

Katz also proved the Frobenius eigenvalues are pure. Our method is not capable of proving this purity result.

(b) *p-adic theorems.* When k is a finite field and $X = \mathbf{G}_{\mathrm{m}}^n \times \mathbf{G}_{\mathrm{a}}^m$, the p-adic properties of the L-series were studied by E. Bombieri [6], and later greatly expanded by A. Adolphson and S. Sperber [1]. The studies of Adolphson and Sperber are based on Dwork's works [16, 17], and methods from singularity theory and toric geometry.

The upshot is that Adolphson and Sperber introduced a complex of p-adic Banach spaces, and an operator α with trace acting on the complex, such that the hyperdeterminant of α gives rise to the L-series of the exponential sum. Moreover, *when the function f is "nondegenerate and convenient"*, Adolphson and Sperber proved that the cohomology spaces of this complex are finite dimensional, and concentrated in a single cohomological degree. The dimensions of these cohomology spaces are the same as the algebraic de Rham cohomology spaces.

Even when the exponential sum *is* defined on $\mathbf{G}_{\mathrm{m}}^n \times \mathbf{G}_{\mathrm{a}}^m$, Theorem 3.15 could imply results that cannot be deduced from the classical theorems, as it could handle Newton-degenerate functions. See Example 4.1 for a (trivial) illustration and Example 4.2 for two more complicated cases.

Remark 0.5. The Dwork–Bombieri–Adolphson–Sperber complex is similar to the complex computing the rigid cohomology (0.1/5), and the former maps into the latter naturally. Their differences can be summarized as follows

- the Dwork–Bombieri–Adolphson–Sperber complex is a complex of Banach spaces, whereas the complex computing rigid cohomology is a complex of ind-Banach spaces, and is never Banach;
- the Dwork–Bombieri–Adolphson–Sperber complex should be thought of as a twisted de Rham complex on a rigid analytic subspace of a toric variety, but the rigid cohomology is defined via a complex on the rigid analytic torus (as a dagger space);
- the finiteness of Dwork–Bombieri–Adolphson–Sperber cohomology does not seem to be known beyond the Newton nondegenerate case (if the exponential sum is defined on a 1-dimensional torus, its finiteness may be recovered using a similar technique we use in these notes); the rigid cohomology of an exponential sum is always finite dimensional (special case of a general theorem of Kedlaya [22]).

We note that a comparison theorem between a dagger variant of Adolphson–Sperber cohomology and rigid cohomology has been proven by Peigen Li [26].

0.6. About the proof. The strategy is to reduce the problems to $\mathbf{A}^1$ via taking direct images, and then use the theory of p-adic ordinary differential equations to deal with the problems on $\mathbf{A}^1$. There are two major inputs, namely Christol and Mebkhout's characterization of "p-adic regular singularity" [8], and Robba's index computation using radii of convergence [29]. It should also be obvious that many of the arguments we present below are influenced by Baldassarri [3], and Chiarellotto [7].

In Section 1 we recall the notion of radius of convergence of a differential module. In Section 2 we explain how to use Robba's index theorem to make local calculations. In Section 3 we globalize the results of Section 2 and prove the main theorem. Section 4 contains some examples. The last section discusses the Higgs variant of Theorem 0.2.

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1. Radii of convergence

This section reviews the notion of radius of convergence of a differential module. We also recall a few basic results, well-known to experts, that we will be using later.

1.1. Notation. In this section we fix the following notation.

- Let K be a complete ultrametric field of characteristic 0. Assume that the residue field of K is of characteristic $p > 0$. Let π be an element of K satisfying $\pi^{p-1} + p = 0$. The field K is the "base field" where spaces are defined.
- Let Ω be an algebraically closed complete ultrametric field containing K, such that $|\Omega| = \mathbf{R}_{\geq 0}$. Assume that the residue field Ω is a transcendental extension of the residue field of K. The field Ω plays an auxiliary role which will give the so-called "generic points" to geometric objects.
- Let I be a connected subset of $\mathbf{R}_{\geq 0}$. Let Δ_I be the rigid analytic space whose underlying set is

$$\{x \in K^{\mathrm{alg}} : |x| \in I\}.$$

Let $\mathbf{D}^{\pm}(x; r)$ be the rigid analytic space whose underlying set is the open/closed disk of radius r centered at $x \in K$. We use $\mathcal{O}$ to denote the sheaf of rigid analytic functions on these spaces.

- In addition to the rigid analytic spaces above, we also consider their extensions to Ω. Let $\Delta_{I,\Omega}$ be the analytic space over Ω whose points are $\{x \in \Omega : |x| \in I\}$. Similarly, for each $\xi \in \Omega$ and $r \in \mathbf{R}_{\geq 0}$, define $\mathbf{D}^+_\Omega(\xi; r) = \{x \in \Omega : |x - \xi| \leq r\}$, and $\mathbf{D}^-_\Omega(\xi, r) = \{x \in \Omega : |x - \xi| < r\}$.
- By a "differential module" over Δ_I or $\mathbf{D}^{\pm}(a; r)$, we shall mean a finite free $\mathcal{O}$-module $\mathcal{E}$ over Δ_I or $\mathbf{D}^{\pm}(a; r)$ equipped with an integrable connection.
- The ρ-Gauss norm on $K[x]$ is

$$\left| \sum_{i \in \mathbf{N}} a_i x^i \right|_\rho = \sup\{|a_i|\rho^i : i \in \mathbf{N}\}.$$

It extends to the field $K(x)$ of rational functions naturally. For $\rho \in \mathbf{R}_{>0}$, denote by F_ρ the completion $K(x)$ with respect to the ρ-Gauss norm $|\cdot|_\rho$. It turns out that F_ρ is also a complete ultrametric field, and carries a continuous extended derivation $\mathrm{d}/\mathrm{d}x$.

- A *differential module* over F_ρ is a finite dimensional F_ρ-vector space V equipped with an K-linear map $D\colon V \to V$, such that for any $a \in F_\rho$, any $v \in V$, the Leibniz rule $D(av) = \frac{\mathrm{d}a}{\mathrm{d}x} v + aD(v)$ holds. It follows that D is automatically continuous.

Remark 1.2 (From $\mathcal{O}$-modules to F_ρ-modules)**.** Let $I \subset \mathbf{R}_{\geq 0}$ be an interval. Let $\rho > 0$ be an element in I. Then there is a natural continuous homomorphism

$$\varphi_\rho\colon \mathcal{O}(\Delta_I) \to F_\rho$$

such that $|\varphi(f)|_\rho = |f|_\rho$.

To construct the homomorphism, first assume $I = [a,b]$ is a closed interval. Then each rigid analytic function f on Δ_I could be written as

$$f(x) = \sum_{n \in \mathbf{Z}} c_n x^n, \quad a_n \in K,$$

such that $c_n b^n \to 0$ as $n \to +\infty$, and $c_n a^n \to 0$ as $n \to -\infty$. In particular, $c_n \rho^n \to 0$ as $n \to \pm\infty$. Set $P_N(x) = \sum_{|n| \leq N} c_n x^n \in K(x)$. Then $P_N \to f$ with respect to the supremum norm of $\mathcal{O}(\Delta_I)$. The condition that $c_n \rho^n \to 0$ implies that $(P_N)_{N=1}^\infty$ is a Cauchy sequence with respect to the ρ-Gauss norm on $K(x)$. Hence $\lim_{N \to +\infty} P_N$ exists in F_ρ. We define this element to be $\varphi_\rho(f)$.

In general, we choose an interval $[a,b] \subset I$ containing ρ and define $\varphi_\rho(f) = \varphi_\rho(f|_{\Delta_{[a,b]}})$. One checks that this definition satisfies the required properties.

Thus, if N is a free $\mathcal{O}$-module on Δ_I for some connected $I \subset \mathbf{R}_{\geq 0}$, $\rho \in I$, then the pullback $V = N \otimes_{\mathcal{O}(\Delta_I), \varphi_\rho} F_\rho$ gives rise to a differential module over F_ρ, with $D(n) = \nabla_{\frac{\mathrm{d}}{\mathrm{d}x}} n$ for any $n \in N$. For simplicity we shall write this tensor product simply by $V = N \otimes_{\mathcal{O}} F_\rho$.

Definition 1.3. Let V be a vector space over F_ρ equipped with a norm $|\cdot|$. Recall that the *operator norm* of an operator T on V is defined to be $|T|_V = \sup_{v \in V - \{0\}} |T(v)|/|v|$; and the *spectral radius* of T to be the quantity

$$|T|_{\mathrm{sp},V} = \lim_{s \to \infty} |T^s|_V^{1/s}.$$

The operator norm of T certainly depends upon the norm, but two equivalent norms determine the same spectral radius [23, Proposition 6.1.5].

Let V be a differential module over F_ρ. Then the *radius of convergence* of V is

$$\mathrm{R}(V) = |\pi| \cdot |D|_{\mathrm{sp},V}^{-1},$$

For $0 < r < 1$, we say a differential module M over $\Delta_{[r,1[}$ or $\Delta_{]r,1[}$ is *overconvergent* (or *solvable* at 1) if $\lim_{\rho \to 1^-} \rho^{-1} \mathrm{R}(M \otimes F_\rho) = 1$.

Example 1.4. The spectral radius of the trivial differential module $(F_\rho, \mathrm{d}/\mathrm{d}x)$ equals $|\pi|\rho^{-1}$. Thus its radius of convergence equals ρ. As the spectral radius of a differential module V is bigger than or equal to that of $\mathrm{d}/\mathrm{d}x$ [23, Lemma 6.2.4], we know that the radius of convergence of any differential module over F_ρ is in the range $]0, \rho]$.

The terminology "radius of convergence" comes from the so-called "Dwork transfer theorem", which we record below.

Theorem 1.5 (Dwork)**.** *Let M be a differential module over Δ_I of rank n. Let $\rho \in I$. Then the following two conditions are equivalent.*

(1) *The radius of convergence of $M \otimes_{\mathcal{O}} F_\rho$ is R.*
(2) *For any $\xi \in \Delta_{\{\rho\},\Omega}$, the restriction of M to the open disk $\mathbf{D}^-_\Omega(\xi; R)$ has n linearly independent horizontal sections.*

Proof. The proof of (1) ⇒ (2) is [23, Theorem 9.6.1]. Here the variable t used by Kedlaya is $t - \xi$ in our context, and the differential module considered by Kedlaya is the restriction of M to the open disk (thus the connection matrix automatically has entries in the ring of analytic elements, fulfilling the hypothesis of the cited theorem).

The proof of (2) ⇒ (1) is [23, Proposition 9.7.5]. Here it is important to note that we should consider the field Ω instead of K itself, so that we have enough "generic points" available. □

The "most" convergent differential modules over Δ_I are said to satisfy the "Robba condition". These modules should be thought of as the correct p-adic analogues of differential modules with regular singularities over a punctured disk.

Definition 1.6. Let M be a differential module over Δ_I. M is said to satisfy the *Robba condition* if for any $\rho \in I$, the differential module $M \otimes_{\mathcal{O}} F_\rho$ has radius of convergence equal to ρ.

Lemma 1.7. *Let M be a differential module over Δ_I satisfying the Robba condition. Then any subquotient differential module of M satisfies the Robba condition.*

Proof. This lemma should be well-known. Let us nevertheless provide the proof for the convenience of the reader.

Let x_0 be a point of $\Delta_{I,\Omega}$. Let M'' be a quotient of M. Then the horizontal basis of $M|_{\mathbf{D}^-_\Omega(x_0;|x_0|)}$ is sent to a set of horizontal sections of M'' which necessarily generate $M''|_{\mathbf{D}^-_\Omega(x_0;|x_0|)}$. This implies that $M''|_{\mathbf{D}^-_\Omega(x_0;|x_0|)}$ is trivial.

Let M' be a differential submodule of M. Now put $M'' = M/M'$. Since

$$\mathrm{H}^0_{\mathrm{DR}}(\mathbf{D}^-_\Omega(x_0; |x_0|), M) \to \mathrm{H}^0_{\mathrm{DR}}(\mathbf{D}^-_\Omega(x_0; |x_0|), M'')$$

is surjective, and since both M and M'' are trivial differential modules over $\mathbf{D}^-_\Omega(x_0; |x_0|)$, the dimension of horizontal sections of M' over $\mathbf{D}^-_\Omega(x_0; |x_0|)$ equals the rank of M', i.e., M' is trivial on $\mathbf{D}^-_\Omega(x_0; |x_0|)$. □

Finally, we quote a theorem due to Christol and Mebkhout [8]. See also [18] and [23, Theorem 13.7.1].

Theorem 1.8 (Christol–Mebkhout)**.** *Let M be a differential module over $\Delta_{]0,1[}$. Assume that there exists a basis $e_1, \dots, e_n$ of M such that*

- *the entries of the matrix representation η of $\nabla_{t\frac{\mathrm{d}}{\mathrm{d}t}}$ with respect to this basis belong to $\mathcal{O}(\mathbf{D}^-(0; 1))$,*
- *M is overconvergent (see Definition 1.3), and*
- *the eigenvalues of $\eta(0)$ belong to $\mathbf{Z}_p \cap \mathbf{Q}$.*

Then there exists a basis of M under which the matrix of $\nabla_{t\frac{\mathrm{d}}{\mathrm{d}t}}$ has entries in $\mathbf{Z}_p$. Moreover, M satisfies the Robba condition.

2. Indices of differential modules

In this section, we use the notion of radii of convergence and Robba's index theorem to prove some cohomology groups are zero. The notation and conventions made in Paragraph 1.1 are still enforced in this section.

Lemma 2.1. *Let N be a differential module over $\Delta_{[a,b]}$. Assume that there exists $a \le \rho \le b$ such that the radius of convergence of any differential submodule of $N \otimes F_\rho$ is $< \rho$. Then $\mathrm{H}^0_{\mathrm{DR}}(\Delta_{[a,b]}, N) = \{0\}$.*

Proof. Let N' be the $\mathcal{O}$-submodule of N generated by horizontal sections of N. Since $\mathcal{O}(\Delta_{[a,b]})$ is noetherian, N' is finitely generated, and is equipped with a trivial connection. It follows that N' is a finite free differential module over $\Delta_{[a,b]}$, say of rank r. Thus $N' \otimes_{\mathcal{O}} F_\rho$ is a trivial differential submodule of $N \otimes_{\mathcal{O}} F_\rho$ of rank r. Being trivial, $N' \otimes_{\mathcal{O}} F_\rho$ has radius of convergence equal to ρ. The hypothesis then implies that $r = 0$, in other words, $N' = 0$ and $\mathrm{H}^0_{\mathrm{DR}}(\Delta_{[a,b]}, N) = \{0\}$. □

We give a simple calculation of the radius of convergence.

Example 2.2. Let L be the differential module on $\Delta_{]0,1[} = \mathbf{D}^-(0;1) - \{0\}$ defined by the system

$$\frac{\mathrm{d}}{\mathrm{d}x} - \frac{\pi}{x^2}.$$

Then for any $0 < \rho < 1$, the radii of convergence of both L and its dual are equal to $\rho^2 < \rho$.

Proof (cf. [29, 5.4.2]). Let $t \in \Omega$ be a any point of radius ρ. Let $x = t + y$. A horizontal section of the differential system is given by $\exp\left(-\pi\left(\frac{1}{t+y} - \frac{1}{t}\right)\right)$. The Taylor series for $\frac{1}{t+y} - \frac{1}{t}$ at $y = 0$ is

$$\sum_{\nu=1}^{\infty} \pm \frac{1}{t^{\nu+1}} y^\nu. \tag{2.3}$$

For each $r < \rho$, the r-Gauss norm of this Taylor series equals

$$\sup\{r^\nu/\rho^{\nu+1} : \nu \in \mathbf{N}\} = r/\rho^2.$$

Thus $\exp\left(\pi\left(\frac{1}{t+y} - \frac{1}{t}\right)\right)$ converges for y in the open disk $\mathbf{D}^-(0;r)$ where $r < \rho^2$.

We claim that $u(y) = \exp\left(\pi\left(\frac{1}{t+y} - \frac{1}{t}\right)\right)$ diverges for some y satisfying $|y| = \rho^2$.

Indeed, write $\frac{1}{t+y} - \frac{1}{t}$ as $\frac{1}{t^2}y + h(y)$. Then $|h(y)|_{\rho^2} < 1$ and $\exp(-\pi h(y))$ is convergent for all $y \in \Omega$ such that $|y| = \rho^2$. It follows that

$$\exp\left(\frac{\pi y}{t^2}\right) = u(y) \cdot \exp(-\pi h(y))$$

is convergent on $\{y \in \Omega : |y| = \rho^2\}$, if $u(y)$ were convergent there. This is absurd, as $\sum \pi^n/n!$ diverges. Thus, the radius of convergence of L equals $\rho^2 < \rho$.

The dual of L is the differential module associated with the differential system

$$\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\pi}{x^2},$$

and the argument is identical. □

Lemma 2.4. *Suppose $[a,b]$ is an interval contained in $]0,1[$. Let M be a differential module on $\Delta_{[a,b]}$ satisfying the Robba condition. Then we have*

$$\mathrm{H}^0_{\mathrm{DR}}(\Delta_{[a,b]}, M\otimes L) = 0.$$

Proof. By Lemma 1.7, any submodule of M satisfies the Robba condition. By [23, Lemma 9.4.6(c)] and Example 2.2, the radii of convergence of all differential submodules of $M\otimes L$ are equal to the radius of convergence of L, which is $<\rho$ at F_ρ. Thus Lemma 2.1 implies the desired result. □

The above vanishing of cohomology groups implies the vanishing of the cohomology groups of some special differential modules over the Robba ring.

Definition 2.5. The *Robba ring* is the colimit

$$\mathcal{R} = \operatorname*{colim}_{r\to 1^-} \mathcal{O}(\Delta_{]r,1[}) = \operatorname*{colim}_{r\to 1^-} \mathcal{O}(\Delta_{[r,1[}).$$

It is equipped with a derivation $\mathrm{d}/\mathrm{d}x$. As in Paragraph 1.1, one can define the notion of a differential module over $\mathcal{R}$. Suppose M is a differential module over $\mathcal{R}$ with derivation D. Define $\mathrm{H}^0_{\mathrm{DR}}(\mathcal{R}, M) = \operatorname{Ker} D$, and $\mathrm{H}^1_{\mathrm{DR}}(\mathcal{R}, M) = \operatorname{Coker} D$.

Lemma 2.6. *Let M be a differential module on the space $\mathbf{D}^-(0;1) - \{0\}$. Assume that M satisfies the hypothesis of Theorem 1.8. Let L be as in Example 2.2. Then we have*

$$\mathrm{H}^0_{\mathrm{DR}}(\mathcal{R}, (M\otimes L)\otimes\mathcal{R}) = \mathrm{H}^1_{\mathrm{DR}}(\mathcal{R}, (M\otimes L)\otimes\mathcal{R}) = 0.$$

Proof. Let s be a horizontal section of $M\otimes L$ over the Robba ring, then there must exist $r<1$ such that the section is defined on the annulus $\Delta_{]r,1[}$, and thus on the annulus $\Delta_{[a,b]}$ for any $[a,b]\subset\,]r,1[$. By Lemma 2.4, we know the section has to be zero on all such $\Delta_{[a,b]}$, and hence the section must be globally zero. This implies that $\mathrm{H}^0_{\mathrm{DR}}(\mathcal{R}, (M\otimes L)\otimes\mathcal{R}) = \{0\}$.

Thanks to the vanishing of H^0, the vanishing of H^1 will follow if we can show the Euler characteristic of $M\otimes L\otimes\mathcal{R}$ is zero. By Theorem 1.8, we can make a $\mathbf{Q}_p$-linear change of bases to put the matrix of $\nabla_{x\frac{\mathrm{d}}{\mathrm{d}x}}$ in a upper triangular form. Thus there is a filtration

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_{-1} = \{0\}$$

of M by differential submodules such that the subquotients M_i/M_{i-1} are of rank 1, necessarily satisfy the Robba condition (Lemma 1.7). Thus, the vanishing of the Euler characteristic of $M\otimes L\otimes\mathcal{R}$ is implied by the vanishing of the Euler characteristic of $M_i/M_{i-1}\otimes L$. Thus we may assume M has rank 1 and is defined by a differential equation $\frac{\mathrm{d}}{\mathrm{d}x} - \frac{c}{x}$, with $c\in\mathbf{Z}_p\cap\mathbf{Q}$.

The de Rham complex for $M\otimes L\otimes\mathcal{R}$ is the filtered colimit

$$\operatorname*{colim}_{r\to 1^-} \mathrm{DR}(M\otimes L\otimes\mathcal{O}_{\Delta_{[r,1[}}).$$

Choose a sequence of numbers $a_n(r)$ such that $a_n(r)\uparrow 1^-$. The above complex reads

$$\operatorname*{colim}_{r\to 1^-} \lim_n \mathrm{DR}(M\otimes L|_{\Delta_{[r,a_n(r)]}}).$$

The transition maps in the inverse system having dense images, one knows that $R^1 \lim_n$ equals zero (Kiehl's Theorem B). Thus it suffices to prove the Euler characteristic of

$$\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}} : M \otimes L|_{\Delta_{[r,a_n(r)]}} \to M \otimes L|_{\Delta_{[r,a_n(r)]}} \tag{2.7}$$

is zero.

Note that $M \otimes L$ is defined by a differential operator of order one with coefficients in $\Omega(x)$. Thus we can use Robba's index theorem [29, Proposition 4.11], which in our situation asserts that, if $I = [a, b]$, then

$$\chi(\nabla_{\mathrm{d}/\mathrm{d}x}) = \frac{\mathrm{d}\log \mathrm{R}((M \otimes L) \otimes_{\mathcal{O}} F_\rho)}{\mathrm{d}\log \rho}\Big|_{\rho=a} - \frac{\mathrm{d}\log \mathrm{R}((M \otimes L) \otimes_{\mathcal{O}} F_\rho)}{\mathrm{d}\log \rho}\Big|_{\rho=b}. \tag{2.8}$$

Since the radius of convergence of $M \otimes L$ at F_ρ always equals ρ^2, both quantities of the right hand side of the displayed equation are equal to 2. Thus the Euler characteristic is zero. This completes the proof of the lemma. □

Lemma 2.9. *Let M be differential module over $\Delta_{]0,1[}$ satisfying the hypothesis of Theorem 1.8. Let L be as in Example 2.2. Then*

$$\mathrm{H}^0_{\mathrm{DR}}(\Delta_{]0,1[}, M \otimes L) = \mathrm{H}^1_{\mathrm{DR}}(\Delta_{]0,1[}, M \otimes L) = \{0\}.$$

Proof. The de Rham cohomology groups are computed by the inverse limit

$$\lim_n \left\{ \Gamma(\Delta_{I_n}, M \otimes L) \xrightarrow{\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}} \Gamma(\Delta_{I_n}, M \otimes L) \right\}$$

where I_n is a sequence of closed intervals $[a_n, b_n]$ contained in $]0, 1[$ such that $a_n \downarrow 0$, $b_n \uparrow 1$. Again, $R^1 \lim$ is zero since the transition maps have dense image (Kiehl's Theorem B). Thus, it suffices to prove the vanishing of $\mathrm{H}^*_{\mathrm{DR}}$ for each Δ_I.

For each closed interval $I \subset]0, 1[$, the vanishing of zeroth cohomology follows from Lemma 2.4. To show that the first cohomology groups are zero, it suffices to prove the Euler characteristic of

$$\Gamma(\Delta_I, M \otimes L) \xrightarrow{\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}} \Gamma(\Delta_I, M \otimes L)$$

is zero. By Theorem 1.8, there is a filtration

$$M = M_n \supset M_{n-1} \supset \cdots \supset M_1 \supset M_{-1} = \{0\}$$

of M by differential submodules such that the subquotients M_i/M_{i-1} are of rank 1, necessarily satisfy the Robba condition. By induction, it suffices to prove the assertion assuming M has rank 1. In the rank 1 case, Robba's index theorem (2.8) applies. Arguing as in the proof of Lemma 2.6 shows that the Euler characteristic is zero. □

3. Rigid cohomology associated with a regular function

3.1. In this section, we continue using the notation made in Section 1. Thus k is a perfect field of characteristic $p > 0$, K is a complete discrete valued field of characteristic 0 containing an element π satisfying $\pi^{p-1} + p = 0$, and the residue field of K is k. The ring of integers of K is denoted by $\mathcal{O}_K$

Our policy is to use Gothic letters to denote schemes over $\mathcal{O}_K$. For a $\mathcal{O}_K$-scheme $\mathfrak{S}$, let $S_0 = \mathfrak{S} \otimes_{\mathcal{O}_K} k$, $S = \mathfrak{S} \otimes_{\mathcal{O}_K} K$. For a finite type K-scheme T, let T^{an} be the rigid analytic space associated with T.

We denote by $\mathbf{D}^-(\infty; r)$ the rigid analytic space $\mathbf{P}^1_K - \mathbf{D}^+(0; r^{-1})$.

3.2. Let conventions be as in Paragraph 3.1. Let $f\colon \overline{\mathfrak{X}} \to \mathbf{P}^1_{\mathcal{O}_K}$ be a proper morphism between smooth $\mathcal{O}_K$-schemes. We make the following assumption.

- $(*)$ Let $S \subset \mathbf{P}^1_K$ be the non-smooth locus of $f\colon \overline{X} \to \mathbf{P}^1_K$. Then the intersection of $\mathbf{P}^1_k$ with the nonsmooth locus of $f\colon \overline{\mathfrak{X}} \to \mathbf{P}^1_{\mathcal{O}_K}$ is contained in the Zariski closure of S. In addition, we assume $S \cap \mathbf{D}^-(\infty; 1)$ is a subset of $\{\infty\}$.

The condition $(*)$ will be used to ensure the Gauss–Manin connection on $D^-(\infty; 1)$ is overconvergent.

Suppose that $\overline{\mathfrak{X}}$ has relative pure dimension n over $\mathcal{O}_K$. Let $\mathfrak{X} = f^{-1}(\mathbf{A}^1_{\mathcal{O}_K})$. Assume that the polar divisor $\mathfrak{P} = f^*(\infty_{\mathcal{O}_K})$ is a relative Cartier divisor with relative strict normal crossings over $\mathcal{O}_K$. Write $\mathfrak{P} = \sum m_i \mathfrak{D}_i$, where $\mathfrak{D}_i$ are smooth proper $\mathcal{O}_K$-schemes. We assume $p \nmid m_i$ for any i. The morphisms $\overline{X}_0 \to \mathbf{P}^1_k$ and $\overline{X} \to \mathbf{P}^1_K$ induced by f are still denoted by f. This abuse of notation is unlikely to cause confusions.

3.3. In order to calculate rigid cohomology, we need to set up some notation for tubular neighborhoods. For $r < 1$, set $[P_0]_{\overline{\mathfrak{X}},r} = f^{-1}(\mathbf{D}^+(\infty; r))$ ("closed tubular neighborhood of radius r"), $]X_0[_{\overline{\mathfrak{X}}} = \overline{X}^{\mathrm{an}} - \bigcup_{r<1} [P_0]_{\overline{\mathfrak{X}},r}$, and $V_r := \overline{X} - [P_0]_{\overline{\mathfrak{X}},r}$.

Denote by j the inclusion map $]X_0[_{\overline{\mathfrak{X}}} \to \overline{X}$, and by j_r the inclusion map $V_r \to \overline{X}$.

3.4 (The Dwork isocrystal)**.** In this paragraph we explain what the Dwork isocrystal is. The affine line $\mathbf{A}^1_k$ sits in the frame $(\mathbf{A}^1_k \subset \mathbf{P}^1_k \subset \widehat{\mathbf{P}}^1_{\mathcal{O}_K})$ where $\widehat{\mathbf{P}}^1_{\mathcal{O}_K}$ is the formal completion of the projective line over $\mathcal{O}_K$ with respect to the maximal ideal of $\mathcal{O}_K$. Therefore to describe this crystal we only need to write down a presentation of it (as an integrable connection) on $\mathbf{P}^{1,\mathrm{an}}_K$.

On the rigid analytic projective line, the tubular neighborhood of $\infty_k \in \mathbf{P}^1_k$ is the complement of the closed unit disk $\mathbf{D}^+(0, 1)$ of radius 1, i.e., $\mathbf{D}^-(\infty; 1)$. The analytification of the algebraic integrable connection

$$\nabla_{\mathrm{D}}\colon \mathcal{O}_{\mathbf{A}^1_K} \to \Omega^1_{\mathbf{A}^1_K}, \quad h \mapsto \mathrm{d}h + \pi h \mathrm{d}t$$

is easily seen to have ρ as its radius of convergence on $D^-(a; \rho)$ for any $|a| \leq 1$, $\rho < 1$. Its restriction to the tubular neighborhood of ∞ is precisely the connection we dealt with in Example 2.2, hence has radius of convergence equal to ρ^2 at F_ρ, which converges to 1 as $\rho \to 1$. Thus $(\mathcal{O}_{\mathbf{A}^{1,\mathrm{an}}_K}, \nabla_{\mathrm{D}})$ is overconvergent for any open disk. By the theory of rigid cohomology ([25, Definition 7.2.14, Proposition 7.2.15]), the differential module determined by $(\mathcal{O}_{\mathbf{A}^{1,\mathrm{an}}_K}, \nabla_{\mathrm{D}})$ gives rise to an overconvergent isocrystal on $\mathbf{A}^1_k$, which could be taken as a coefficient system for the rigid cohomology. We denote it by $\mathcal{L}_\pi$, and call it the Dwork isocrystal [25, §4.2.1].

3.5. Let notation and conventions be as in Paragraph 3.2. Consider the morphism $f\colon X \to \mathbf{A}^1_K$. Then we can define an algebraic integrable connection on the structure sheaf $\mathcal{O}_X$

$$\nabla_{\pi f}\colon \mathcal{O}_X \to \Omega^1_{X/K}, \quad \nabla_{\pi f}(h) = \mathrm{d}h + \pi h \mathrm{d}f.$$

This integrable connection is the inverse image of the connection $(\mathcal{O}_{\mathbf{A}^1}, \nabla_{\mathrm{D}})$ via the morphism f.

By analytification, we obtain an analytic connection, still denoted by $\nabla_{\pi f}$, on the rigid analytic space X^{an}.

Proposition 3.6. *Let notation and conventions be as in Paragraphs 3.2 – 3.5. Then for each integer m, the natural maps*

$$\mathrm{H}^m_{\mathrm{DR}}(X, \nabla_{\pi f}) \to \mathrm{H}^m_{\mathrm{DR}}(X^{\mathrm{an}}, \nabla_{\pi f}) \to \mathrm{H}^m_{\mathrm{rig}}(X_0/K, f^*\mathcal{L}_\pi)$$

are isomorphisms of K-vector spaces.

Proof that the right hand side arrow is an isomorphism. Without loss of generality we could assume X is irreducible. When $f|_X \colon X \to \mathbf{A}^1_K$ is a constant morphism, the proposition is trivial. In the sequel we shall assume $f|_X \colon X \to \mathbf{A}^1_K$ is surjective.

To begin with, let us write down the complex that computes the rigid cohomology. Let

$$j^\dagger \Omega^i_{\overline{X}^{\mathrm{an}}/K} = \operatorname*{colim}_{r \to 1^-} j_{r*} j_r^* \Omega^i_{\overline{X}^{\mathrm{an}}/K}.$$

The analytification of the connection $\nabla_{\pi f}$ (still denoted by $\nabla_{\pi f}$) gives rise to an integrable connection

$$\nabla^\dagger \colon j^\dagger \mathcal{O}_{\overline{X}^{\mathrm{an}}} \to j^\dagger \Omega^1_{\overline{X}^{\mathrm{an}}},$$

which extends to a dagger version of the de Rham complex

$$\mathcal{DR}(\overline{X}^{\mathrm{an}}, \nabla^\dagger) : \qquad j^\dagger \mathcal{O}_{\overline{X}^{\mathrm{an}}} \xrightarrow{\nabla^\dagger} j^\dagger \Omega^1_{\overline{X}^{\mathrm{an}}} \xrightarrow{\nabla^\dagger} \cdots \xrightarrow{\nabla^\dagger} j^\dagger \Omega^n_{\overline{X}^{\mathrm{an}}}.$$

For an admissible open subspace V of $\overline{X}^{\mathrm{an}}$, let $\mathcal{DR}(V, \nabla_{\pi f}|_V)$ be the de Rham complex of $\nabla_{\pi f}$ restricted to V. Set $\mathrm{DR}(V, \nabla_{\pi f}|_V) = R\Gamma(V, \mathcal{DR}(V, \nabla_{\pi f}|_V))$. We have

$$\mathrm{DR}(\overline{X}^{\mathrm{an}}, \nabla^\dagger) := R\Gamma(\overline{X}^{\mathrm{an}}, \mathcal{DR}(X^{\mathrm{an}}, \nabla^\dagger)) = \operatorname*{colim}_{r \to 1} R\Gamma(\overline{X}^{\mathrm{an}}, Rj_{r*}\mathcal{DR}(V_r, \nabla_{\pi f}|_{V_r})).$$

A priori, this complex depends upon the formal completion of the scheme $\overline{\mathfrak{X}}$ along the maximal ideal of $\mathcal{O}_K$. However, since $\overline{\mathfrak{X}}$ is proper and the integrable connection $(\mathcal{O}_{X^{\mathrm{an}}}, \nabla_{\pi f})$ is overconvergent (being the inverse image of an overconvergent integrable connection), [25, Corollary 8.2.3] asserts that it only depends upon X_0 and $f|_{X_0} \colon X_0 \to \mathbf{A}^1_k$. We have then

$$\mathrm{DR}(\overline{X}^{\mathrm{an}}, \nabla^\dagger) = R\Gamma_{\mathrm{rig}}(X_0, f_0^*\mathcal{L}_\pi).$$

Next we explain how to compute the cohomology of the de Rham complex. For any r sufficiently close to 1, set $V_r = f^{-1}(\mathbf{D}^-(0; r^{-1}))$. Then X^{an} has an admissible open covering

$$X^{\mathrm{an}} = V_r \cup f^{-1}(\mathbf{D}^-(\infty; 1) - \{\infty\}).$$

In the bounded derived category of K-vector spaces, the Mayer–Vietoris theorem gives an isomorphism between the de Rham complex $\mathrm{DR}(X, \nabla_{\pi f})$ and the homotopy fiber of the map

$$\mathrm{DR}(V_r, \nabla_{\pi f}) \oplus \mathrm{DR}(f^{-1}(\mathbf{D}^-(\infty; 1) - \{\infty\}), \nabla_{\pi f}) \to \mathrm{DR}(V_r \cap f^{-1}(\mathbf{D}^-(\infty; 1)), \nabla_{\pi f}).$$

Since the colimit, as $r \to 1^-$, of $\mathrm{DR}(V_r, \nabla_{\pi f})$ equals $R\Gamma_{\mathrm{rig}}(X_0, f_0^*\mathcal{L}_\pi)$, in order to prove the comparison between rigid and de Rham cohomology it suffices to prove the natural morphism

(3.7)
$$\mathrm{DR}(f^{-1}(\mathbf{D}^-(\infty;1) - \{\infty\}), \nabla_{\pi f}) \to \operatorname*{colim}_{r\to 1^-} \mathrm{DR}(V_r \cap f^{-1}(\mathbf{D}^-(\infty;1) - \{\infty\}), \nabla_{\pi f})$$

is an isomorphism in the derived category of vector spaces over K.

Let S be the finite subspace of $\mathbf{A}_K^{1,\mathrm{an}}$ containing all the critical values of f. Let $X' = X - f^{-1}(S)$. Then the direct image sheaf

$$\mathcal{E} = R^i f_*(\Omega^\bullet_{X'/K}, \mathrm{d}), \tag{3.8}$$

is equipped with a Gauss–Manin connection ∇_{GM}. By projection formula, we have $R^i(f|_{X'})_*(\mathcal{DR}(\mathcal{O}_{X^{\mathrm{an}}}, \nabla_{\pi f})|_{X'})$ is isomorphic to the analytification of the tensor product

$$\mathcal{E} \otimes (\mathcal{O}_{\mathbf{A}^1_K}, \nabla_{\mathrm{D}})$$

on $\mathbf{A}_K^{1,\mathrm{an}} - S$.

Thus, if $u \colon \mathbf{P}_K^{1,\mathrm{an}} -]S_0[_{\mathbf{P}^1_{\mathcal{O}_K}} \to \mathbf{P}_K^{1,\mathrm{an}}$ denotes the open immersion, then

$$u^\dagger(\mathcal{E}^{\mathrm{an}}, \nabla_{\mathrm{GM}}),$$

is the rigid cohomological direct image $R^i(f|_{X_0'})_{\mathrm{rig}*}\mathcal{O}_{X_0'}$.

Using the Leray spectral sequence, we see that in order to prove the morphism (3.7) is an isomorphism, it suffices to prove that for any i, the map

(3.9)
$$R\Gamma(\mathbf{D}^-(\infty;1) - \{\infty\}, \mathcal{E}^{\mathrm{an}} \otimes (\mathcal{O}_{\mathbf{A}_K^{1,\mathrm{an}}}, \nabla_{\mathrm{D}})) \to \operatorname*{colim}_{r\to 1^-} R\Gamma(\Delta_{]1,r^{-1}]}, \mathcal{E}^{\mathrm{an}} \otimes (\mathcal{O}_{\mathbf{A}_K^{1,\mathrm{an}}}, \nabla_{\mathrm{D}}))$$

is an isomorphism.

We shall show that both sides of (3.9) are acyclic. For convenience, we shall use a coordinate x around $\infty \in \mathbf{P}_K^{1,\mathrm{an}}$, thus swap ∞ and 0. Let M be the restriction of the analytification of $\mathcal{E} = R^i f_*(\Omega^\bullet_{X'/K})$ to the disk $\mathbf{D}^-(0;1) - \{0\}$. In Lemma 3.10 below, we shall explain that M satisfies the hypothesis of Theorem 1.8. On the other hand, the differential module L considered in Example 2.2 is precisely the restriction of $(\mathcal{O}_{\mathbf{A}^1_K}, \nabla_{\mathrm{D}})^{\mathrm{an}}$ in the vicinity of ∞.

The right hand side of (3.9) now reads

$$\operatorname*{colim}_{r\to 1^-} \left\{ (M\otimes L)|_{\Delta_{[r,1[}} \xrightarrow{\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}} (M\otimes L)|_{\Delta_{[r,1[}} \right\},$$

which is precisely the de Rham complex of $M \otimes L$ restricted to the Robba ring:

$$(M\otimes L) \otimes_{\mathcal{O}} \mathcal{R} \xrightarrow{D} (M \otimes L) \otimes_{\mathcal{O}} \mathcal{R}.$$

Thus, Lemma 2.6 implies that the right hand side of (3.9) is trivial. The acyclicity of the left hand side of (3.9) follows from Lemma 2.9. This completes the proof that the analytic twisted de Rham cohomology is isomorphic to the rigid cohomology. □

Lemma 3.10. *Let notation be as above. Then the differential module M satisfies the hypotheses of Theorem 1.8.*

Proof. By the regularity of the Gauss–Manin system, the differential module M is a restriction of an algebraic integrable connection which is regular singular around 0. In particular, there exists a basis of M such that the derivation $\nabla_{\frac{\mathrm{d}}{\mathrm{d}x}}$ is given by

$$\frac{\mathrm{d}}{\mathrm{d}x} - \eta(x)$$

where η is a rational function which has at worst a simple pole at $x = 0$ (for example, take the restriction of an algebraic basis and restrict to the analytic open $\mathbf{D}^{-}(\infty; 1)$).

As we have assumed that the multiplicities of the polar divisor are prime to p, the algebraic calculation of exponents of the Gauss–Manin system around infinity implies that the eigenvalues of $(x\eta)|_{x=0}$ belong to $\mathbf{Z}_p \cap \mathbf{Q}$. (One can embed field of definition of the variety into $\mathbf{C}$, then use [12, Exposé XIV, Proof of Proposition 4.15] to show that the eigenvalue of $\eta(0)$ with respect to the algebraic basis are rational numbers whose denominators are not divisible by p.)

Finally the overconvergence of the Gauss–Manin system in the lifted situation under the hypothesis $(*)$ is proved by P. Berthelot [5, Théorème 5] and N. Tsuzuki [32, Theorem 4.1.1]. □

Proof that the left hand side arrow is an isomorphism. This is a theorem of André and Baldassarri [2, Theorem 6.1]. Since the precise hypothesis of their theorem is not met in the present situation (the connection is not defined over a number field), we shall nevertheless provide a proof. The key point is a theorem of Clark, which states that in a good situation, the analytic index of a differential operator equals its formal index.

By GAGA, the algebraic de Rham complex is computed by complex

$$\mathcal{DR}_{\mathrm{mer}}\colon \quad \mathcal{O}_{\overline{X}^{\mathrm{an}}}(*P_{\mathrm{red}}) \xrightarrow{\nabla_{\pi f}} \Omega^1_{\overline{X}^{\mathrm{an}}}(*P_{\mathrm{red}}) \to \cdots \to \Omega^n_{\overline{X}^{\mathrm{an}}}(*P_{\mathrm{red}}),$$

which is a subcomplex of $\mathcal{DR}(X^{\mathrm{an}}, \nabla_{\pi f})$. Here $\Omega^m_{\overline{X}^{\mathrm{an}}}(*P_{\mathrm{red}}) = \bigcup_{e=1}^{\infty} \Omega^m_{\overline{X}^{\mathrm{an}}}(eP_{\mathrm{red}})$. Cover $\overline{X}^{\mathrm{an}}$ by $X^{\mathrm{an}} = f^{-1}(\mathbf{A}^{1,\mathrm{an}})$ and $[P_0]_{\overline{\mathfrak{X}},\epsilon} = f^{-1}(\mathbf{D}^{+}(\infty;\epsilon))$, the Mayer–Vietoris theorem implies that $\mathrm{DR}(X, \nabla_{\pi f})$ is the homotopy fiber of

$$\mathrm{DR}(X^{\mathrm{an}}, \nabla_{\pi f}) \oplus R\Gamma(f^{-1}(\mathbf{D}^{+}(\infty;\epsilon)), \mathcal{DR}_{\mathrm{mer}}) \to \mathrm{DR}(X^{\mathrm{an}} \cap f^{-1}(\mathbf{D}^{+}(\infty;\epsilon)), \nabla_{\pi f}).$$

Taking colimit with respect to $\epsilon \to 0$, we see it suffices to prove the colimit, as $\epsilon \to 0$, of the following map

$$R\Gamma(f^{-1}(\mathbf{D}^{+}(\infty;\epsilon)), \mathcal{DR}_{\mathrm{mer}}) \to \mathrm{DR}(X^{\mathrm{an}} \cap f^{-1}(\mathbf{D}^{+}(\infty;\epsilon)), \nabla_{\pi f}).$$

is a quasi-isomorphism. Again, we shall show both items are acyclic.

Let $\mathcal{E}$ be the algebraic Gauss–Manin system on $\mathbf{A}^1 - S$ as in (3.8). Let $\iota\colon \mathbf{A}^1_K - S \to \mathbf{P}^1_K$ be the inclusion. Let $\mathcal{E}(*S) = \iota_* \mathcal{E}$. Using Leray spectral sequence, it suffices to prove the de Rham complex of

$$\operatorname*{colim}_{\epsilon \to 0} (\mathcal{E}^{\mathrm{an}} \otimes (\mathcal{O}_{\mathbf{A}^{1,\mathrm{an}}_K}, \nabla_{\mathrm{D}}))|_{\mathbf{D}^{+}(\infty;\epsilon) - \{\infty\}} \tag{3.11}$$

("de Rham complex with essential singularities") and that of

$$\operatorname*{colim}_{\epsilon \to 0} (\mathcal{E}(*S) \otimes (\mathcal{O}_{\mathbf{A}^1_K}(*S), \nabla_{\mathrm{D}}))^{\mathrm{an}}|_{\mathbf{D}^{+}(\infty;\epsilon)}. \tag{3.12}$$

("de Rham complex with moderate singularities") are acyclic.

A similar argument as in the proof of Lemma 2.9 using Robba's index theorem yields that (3.11) has zero Euler characteristic and vanishing H^0, thus acyclic.

Since the de Rham complex of (3.12) is a subcomplex of that of (3.11), its H^0 is also trivial.

We proceed to prove that (3.12) has zero Euler characteristic. Let $\mathcal{O}_0$ be the local ring $\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_K,\infty}$ with the uniformizer x defined by a coordinate around ∞. Let $\widehat{\mathcal{O}}_0 \cong K[\![x]\!]$ be its x-adic completion. Then the de Rham complex of (3.12) is that of

$$\begin{aligned}&(\mathcal{E}(*S)\otimes(\mathcal{O}_{\mathbf{A}^1_K}(*S),\nabla_{\mathrm{D}}))^{\mathrm{an}}\otimes_{\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_K}}\mathcal{O}_0\\ =&(\mathcal{E}(*S)\otimes(\mathcal{O}_{\mathbf{A}^1_K}(*S),\nabla_{\mathrm{D}}))^{\mathrm{an}}\otimes_{\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_K}}\mathcal{O}_0[1/x].\end{aligned}$$

Choose a cyclic vector [11, II 1.3] for the differential module

$$(\mathcal{E}(*S)\otimes(\mathcal{O}_{\mathbf{A}^1_K}(*S),\nabla_{\mathrm{D}}))^{\mathrm{an}}\otimes_{\mathcal{O}_{\mathbf{P}^{1,\mathrm{an}}_K}}\mathcal{O}_0[1/x]$$

over the differential field $\mathcal{O}_0[1/x]$. Thus we obtain a differential operator $u = \sum_i a_i \frac{\mathrm{d}^i}{\mathrm{d}x^i}$ with $a_i \in \mathcal{O}_0$, and the Euler characteristic equals the index of

$$\mathcal{O}_0[1/x] \xrightarrow{u} \mathcal{O}_0[1/x].$$

On the other hand, Malgrange's index theorem [27, Théorème 2.1b] implies that the index of

$$\widehat{\mathcal{O}}_0[1/x] \xrightarrow{u} \widehat{\mathcal{O}}_0[1/x]$$

equals zero. Thus, it suffices to prove the index of

$$\widehat{\mathcal{O}}_0/\mathcal{O}_0 \xrightarrow{u} \widehat{\mathcal{O}}_0/\mathcal{O}_0 \tag{3.13}$$

equals zero. Since $\mathcal{E}$ is regular singular, ∇_{D} is rank one and irregular, the indicial polynomial of u is zero. Thus the hypothesis on non-Liouville difference in Clark's theorem [9] as stated in [7, Théorème 2.12] is satisfied, and this theorem implies the index of (3.13) is zero. This completes the proof of Proposition 3.6. □

Next we prove the main result of these notes by removing the properness hypothesis from Proposition 3.6.

3.14. We follow the conventions made in Paragraph 3.1. Let $f\colon \overline{\mathfrak{X}} \to \mathbf{P}^1_{\mathcal{O}_K}$ be a proper morphism between smooth $\mathcal{O}_K$-schemes, such that the condition $(*)$ in Paragraph 3.2 is satisfied. Assume $\overline{\mathfrak{X}}$ has relative pure dimension n over $\mathcal{O}_K$.

Let $\mathfrak{X}$ be a Zariski open subscheme of $\overline{\mathfrak{X}}$, such that

- $\mathfrak{X} \subset \mathfrak{Y} := f^{-1}(\mathbf{A}^1_{\mathcal{O}_K})$,
- $\mathfrak{D} = \overline{\mathfrak{X}} - \mathfrak{X}$ is a relative Cartier divisor with relative strict normal crossings over $\mathcal{O}_K$.

The support of the polar divisor $\mathfrak{P} = f^*(\infty_{\mathcal{O}_K})$ is therefore contained in $\mathfrak{D}$. We could write $\mathfrak{P} = \sum_{i=1}^r m_i \mathfrak{D}_i$. Again, we assume $p \nmid m_i$ for any i. Finally we could write $\mathfrak{D} = \sum_{i=1}^r \mathfrak{D}_i + \sum_{j=1}^s \mathfrak{H}_i$. Thus, for any subset I of $\{1,2,\ldots,r\}$ and any subset J of $\{1,2,\ldots,s\}$, the intersection

$$\bigcap_{i\in I}\mathfrak{D}_i \cap \bigcap_{j\in J}\mathfrak{H}_j$$

is a smooth proper $\mathcal{O}_K$-scheme.

The following theorem is a more general version of Proposition 3.6: when $\mathfrak{P}_{\mathrm{red}} = \mathfrak{D}$, the theorem reduces to Proposition 3.6; the general case is deduced from Proposition 3.6 by some standard yoga.

Theorem 3.15. *Let notation and conventions be as in Paragraph 3.14. Then for each integer m, the natural maps*

$$\mathrm{H}^m_{\mathrm{DR}}(X, \nabla_{\pi f}) \to \mathrm{H}^m_{\mathrm{DR}}(X^{\mathrm{an}}, \nabla_{\pi f}) \to \mathrm{H}^m_{\mathrm{rig}}(X_0/K, f^*\mathcal{L}_\pi)$$

are isomorphisms.

Proof. For each subset J of $\{1, 2, \ldots, s\}$, denote by $\mathfrak{H}_J$ the intersection $\bigcap_{j\in J} \mathfrak{H}_j$ and denote by $\mathfrak{H}_J^\circ$ the intersection $\left(\bigcap_{j\in J} \mathfrak{H}_j\right) \cap \mathfrak{Y}$. (Convention: when J is the empty set, $\mathfrak{H}_J^\circ$ is understood as $\mathfrak{Y}$.) Then each $\mathfrak{H}_J$ is a smooth proper $\mathcal{O}_K$-scheme, and the restriction of f to $\mathfrak{H}_J$ is denoted by f_J, which is a proper morphism into $\mathbf{P}^1_{\mathcal{O}_K}$. The polar divisor of f_J is the restriction of $\mathfrak{P}$ to $\mathfrak{H}_J$. Note that the restriction of $f^*\nabla_\pi$ to H_J° is none other than $f_J^*\nabla_\pi$.

By Proposition 3.6, for each J, the natural maps

$$\mathrm{H}^m_{\mathrm{DR}}(H_J^\circ, \nabla_{\pi f_J}) \to \mathrm{H}^m_{\mathrm{DR}}(H_J^{\circ,\mathrm{an}}, \nabla_{\pi f_J}) \to \mathrm{H}^m_{\mathrm{rig}}((H_J^\circ)_0/K, f_J^*\mathcal{L}_\pi)$$

are isomorphisms.

There exists a second-quadrant spectral sequence with

$$E_1^{-i,j} = \bigoplus_{\mathrm{Card}\, J = i} \mathrm{H}^{j-2i}_{\mathrm{rig}}((H_J^\circ)_0/K, f_J^*\mathcal{L}_\pi) \tag{3.16}$$

$(i, j \geq 0)$ which abuts to $\mathrm{H}^{-i+j}_{\mathrm{rig}}(X_0/K, f^*\mathcal{L}_\pi)$. Indeed, Mayer–Vietoris for a finite closed covering [25, Proposition 7.4.13] gives a spectral sequence

$$E_1^{a,b} = \bigoplus_{\mathrm{Card}\, J = a+1} \mathrm{H}^b_{\mathrm{rig,c}}((H_J^\circ)_0/K, f_J^*\mathcal{L}_\pi) \Rightarrow R\Gamma_{\mathrm{rig,c}}\left(\bigcup_{j=1}^s (H_j^\circ)_0/K, f^*\mathcal{L}_\pi|_{\bigcup (H_j^\circ)_0}\right),$$

$(a, b \geq 0)$. Using the normal crossing hypothesis, applying Poincaré duality [25, Corollary 8.3.14], the finiteness of rigid cohomology [22], and replacing π by $-\pi$ (since the dual of $\mathcal{L}_\pi$ is $\mathcal{L}_{-\pi}$), we obtain a spectral sequence

$$E_1^{-a,b} = \bigoplus_{\mathrm{Card}\, J = a+1} \mathrm{H}^{b-2a}_{\mathrm{rig}}((H_J^\circ)_0/K, f_J^*\mathcal{L}_\pi) \Rightarrow R\Gamma_{\mathrm{rig}, \bigcup (H_j^\circ)_0}(Y_0/K, f^*\mathcal{L}_\pi) \tag{3.17}$$

$(a, b \geq 0)$. Since $R\Gamma_{\mathrm{rig}}(X_0/K, \bullet)$ fits into a distinguished triangle

$$R\Gamma_{\mathrm{rig}, \bigcup (H_j^\circ)_0}(Y_0/K, \bullet) \to R\Gamma_{\mathrm{rig}}(Y_0/K, \bullet) \to R\Gamma_{\mathrm{rig}}(X_0/K, \bullet) \to$$

(the derived incarnation of the relative cohomology sequence), we deduce the desired spectral sequence (3.16) by augmenting the rigid cohomology of Y_0 to the zeroth column of the spectral sequence (3.17).

We have similar spectral sequences for the algebraic and analytic de Rham cohomology groups. The E_1-differentials of these spectral sequence are all Gysin maps in the various theories. Thus the natural maps between the theories give rise to maps of these E_1-spectral sequences. Proposition 3.6 implies that these maps are isomorphisms on the E_1-stage. Thus they induce isomorphisms on the abutments. □

Proof of Theorem 0.2. Let notation be as in the statement of Theorem 0.2. Let $\mathbf{K}$ be the field of fractions of R. Using resolution of singularities, upon making a finite extension of $\mathbf{K}$ (both rigid and de Rham cohomology are compatible with extensions, so performing a finite extension does not change the result), we can embed $X_{\mathbf{K}}$ into a smooth proper $\mathbf{K}$-scheme $\overline{X}_{\mathbf{K}}$ such that $\overline{X}_{\mathbf{K}} - X_{\mathbf{K}}$ has strict normal crossings, and f extends to a morphism $f\colon \overline{X}_{\mathbf{K}} \to \mathbf{P}^1_{\mathbf{K}}$. There is a Zariski open subset U of $\mathrm{Spec}(R)$ over which

- $\overline{X}_{\mathbf{K}}$ as well as all the intersections of the boundary divisors have good reduction at primes in U, and
- the multiplicities of the polar divisor are not divisible by residue characteristics of U.

We may assume U is affine, and by abusing notation still denote its associated ring by R. For each closed point $\mathfrak{p}$ in U, let p be the characteristic of the residue field $\kappa(\mathfrak{p})$. Then there exists a p-adically complete discrete valuation ring R''' containing ζ_p and a ring homomorphism $R \to R'$, such that the closed point of $\mathrm{Spec}(R')$ is mapped to $\mathfrak{p}$. Let K' be the field of fractions of R' and let k' be its residue field. Then we have

$$\mathrm{H}^{\bullet}_{\mathrm{rig}}(X_{\kappa(\mathfrak{p})}/K_{\mathfrak{p}}, \mathcal{L}_\pi) \otimes_{K_{\mathfrak{p}}} K' \cong \mathrm{H}^{\bullet}_{\mathrm{rig}}(X_{k'}/K', \mathcal{L}_\pi).$$

We can then apply Theorem 3.15 to $X_{R'}$, obtaining an isomorphism between the rigid cohomology $\mathrm{H}^{\bullet}_{\mathrm{rig}}(X_{k'}/K', \mathcal{L}_\pi)$ and a twisted de Rham cohomology of $X_{K'}$.

To conclude, we use the following two facts: (i) ∇_{cf} and ∇_f have isomorphic de Rham cohomology group over K', for any $c \in K'^{\times}$; and (ii) the formation of twisted algebraic de Rham cohomology is compatible with extension of scalars. The fact (i) is proved below as a lemma. □

Lemma 3.18. *Let K be a field of characteristic 0. Let $f\colon X \to \mathbf{A}^1$ be a morphism of smooth K-schemes. Then for all $c \in K^{\times}$, the dimensions of the K-vector spaces $\mathrm{H}^i(X, \nabla_{cf})$ are the same.*

Proof. It suffices to prove ∇_{cf} and ∇_f have isomorphic twisted de Rham cohomology groups. The standard argument (extracting coefficients defining X and f, choosing a finitely generated subfield, embedding the field into $\mathbf{C}$) allows us to assume $K = \mathbf{C}$. Then we can use the isomorphism provided by (0.1/1) and (0.1/2) to conclude that the two twisted de Rham cohomology groups are isomorphic to the relative cohomology groups

$$\mathrm{H}^i(X^{\mathrm{an}}, f^{-1}(t)^{\mathrm{an}}), \quad \text{and} \quad \mathrm{H}^i(X^{\mathrm{an}}, (cf)^{-1}(t)^{\mathrm{an}}) \quad (|t| \gg 0)$$

respectively. When $|t|$ is large, these two groups are isomorphic, since f is topologically a fibration away from finitely many points. □

4. Examples

Example 4.1. As a sanity check, here is a trivial example calculating the degree of the L-series of a Newton degenerate polynomial.

Let k be a finite field of characteristic > 2. Let $f\colon \mathbf{A}^2_k \to \mathbf{A}^1_k$ be the regular function defined by the polynomial $f(x, y) = x^2y - x$. (The projective completion of the curve $f(x,y)=t$, $t\neq 0$, has a cuspidal singularity at $[0,1,0]$, which can be simultaneously resolved by blowing this point up.)

The morphism is smooth, the polynomial f is Newton nondegenerate, but it is not convenient. Thus the theorems of Adolphson–Sperber [1] and Denef–Loeser [14] do not give the degree of the L-series.

A direct computation is easy:

$$\begin{aligned} S_m(f) &= \sum_{x,y\in k_m} \psi_m(x^2y-x)) \\ &= \sum_{x,y\in k_m} \psi_m(x^2y)\psi_m(x)^{-1} \\ &= q^m + \sum_{x\neq 0}\psi(x)^{-1}\sum_{y\in k_m}\psi_m(x^2y) \\ &= q^m. \end{aligned}$$

It follows that the L-series is $(1-qt)^{-1}$.

Here is a topological calculation. It is easy to see $f(x,y)=t$ for $t\neq 0$ is isomorphic to $\mathbf{G}_{\mathrm{m}}$ as schemes over k. Thus one can still use the Teichmüller lift of f to conclude that the rigid cohomology $\mathrm{H}^*(\mathbf{A}^2_k, f^*\mathcal{L}_\pi)$ is 1-dimensional in degree 2, and zero otherwise. In particular, $L_f(t)^{-1}$ is a linear polynomial.

Next, we recall the degree and total degree of the L-series of exponential sums. Let notation be as in Paragraph (0.1/4). In view of Grothendieck's theorem, we can write $L_f(t)=P(t)/Q(t)$, where $P,Q\in\overline{\mathbf{Q}}[t]$ are coprime. Then the *degree* of L_f is $\deg Q-\deg P$; the *total degree* of L_f is $\deg P+\deg Q$.

By the trace formula for rigid cohomology, we know that the degree of L_f is equal to the Euler characteristic $\sum(-1)^i\dim\mathrm{H}^i_{\mathrm{rig,c}}(X,f^*\mathcal{L}_\pi)$, and the total degree of L_f is no bigger than $\sum\dim\mathrm{H}^i_{\mathrm{rig,c}}(X,f^*\mathcal{L}_\pi)$. (Since X is smooth, Poincaré duality identifies the dimension of the compactly supported cohomology with the dimension of the rigid cohomology up to a dimension shift.)

Example 4.2. Hyperplane arrangements give rise to many interesting exponential sums whose L-series cannot be computed using traditional methods. But topological and combinatorial methods sometimes can deduce useful information about the Milnor fiber of the arrangement, which, through Theorem 3.15, can determine the dimension of the rigid cohomology for large primes.

Here are two concrete examples.

(1) Let $f(x,y,z)=xyz(x-y)(y-z)(z-x)$ be the polynomial defining the so-called A_3 plane arrangement. Then f is not convenient, but Cohen and Suciu [10, Example 5.1] have shown that the Betti numbers of the "Milnor fiber" $f=1$ are: $b_0=1$, $b_1=7$, $b_2=18$. Since f is homogeneous, all the fibers $f^{-1}(t)$, $t\neq 0$, are homeomorphic. It follows that the relative cohomology $\mathrm{H}^\bullet(\mathbf{C}^3,f^{-1}(t))$ equals the *reduced* singular cohomology of $f^{-1}(1)$, up to a shift of cohomology degree. Using the isomorphism between relative cohomology (0.1/1) and twisted de Rham cohomology (0.1/2), we deduce that the nonzero Betti numbers of the twisted de Rham cohomology are $b_2=7, b_3=18$. For p large, Theorem 0.2 implies that the rigid cohomology groups of $f^*\mathcal{L}_\pi$ with compact support are of dimensions

$0, 7, 18$ respectively. In particular, the degree of the L-series of the exponential sum associated with f has degree 11; and total degree ≤ 25.

(2) Let $f(x, y, z) = xyz(x+y)(x-y)(x+z)(x-z)(y+z)(y-z)$ be the polynomial defining the so-called B_3 plane arrangement. Cohen and Suciu [10, Example 5.2] have shown that the Betti numbers of the Milnor fiber $f = 1$ are: $b_0 = 1$, $b_1 = 8$, $b_2 = 79$. As in (1), when p is large, we get the dimension of the rigid cohomology of $f^*\mathcal{L}_\pi$. We may conclude that the degree of the L-series of f is 71, and the total degree is ≤ 87.

Example 4.3 (Exponential sum on a curve)**.** Let X be a smooth irreducible projective curve of genus g over a finite field k. Let $f\colon X \to \mathbf{P}^1_k$ be a generically étale, degree d morphism with ramification indices not divisible by p. We assume there is a good lift of f (i.e., satisfies Hypothesis 3.14). Let $Z = f^{-1}(\infty) = \{\tau_1, \dots, \tau_m\}$. Let V be a nonempty Zariski open subset of X such that $f(V) \subset \mathbf{A}^1_k$. We assume that the ramification points of f as well as the points $X - V$ are rational over k (which is harmless in considering the degree of L-series). Let $c = \mathrm{Card}(Z - V)$.

Then a topological calculation implies that the twisted algebraic de Rham cohomology of a lift of f has cohomology in degree 1 only, and the dimension equals $2g + c + m + d - 2$. In this case Theorem 0.2 applies and we conclude that $L^{-1}_{f|_C}(t)$ is a polynomial of degree $2g + c + m + d - 2$. This matches the length of the "Hodge polygon" considered by Kramer-Miller [24, Theorem 1.1].

Example 4.4 (Exponential sum on SL_2)**.** Let k be a finite field of characteristic $p > 0$. Let $V = k^2$ be the standard representation of SL_2. For $A \in \mathrm{SL}_2(k)$, let $A^{(n)} = \mathrm{Tr}(\mathrm{Sym}^n A)$. Then for $a_1, \dots, a_N \in k$,

$$f(A) = \sum_{n=1}^{N} a_n A^{(n)}$$

is a regular function on SL_2. Then the rigid cohomology $\mathrm{H}^*_{\mathrm{rig}}(\mathrm{SL}_2, f^*\mathcal{L}_\pi)$ is related to the exponential sum

$$\sum_{A \in \mathrm{SL}_2(k_m)} \psi_m(f(A)), \tag{4.5}$$

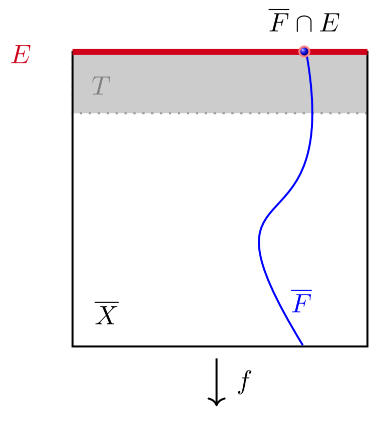
where k_m is a fixed degree m extension of k, ψ_1 is a nontrivial additive character on k, $\psi_m = \psi_1 \circ \mathrm{Tr}_{k_m/k}$.

If p is sufficiently large, and if $(a_1, \dots, a_N)$ is sufficiently general, we can calculate the dimension of the rigid cohomology using topology. It is not difficult to see $f^{-1}(t)$ is N disjoint union of $\mathbf{P}^1 \times \mathbf{P}^1 - \Delta$. Using the long exact sequence for relative cohomology one sees that $\mathrm{H}^*(\mathrm{SL}_2(\mathbf{C}), f^{-1}(t))$ is nonzero only in degree 1, and 3 and of dimension $N - 1$, $N + 1$ respectively. Thus the L-series $L_f(t)$ of the exponential sums (4.5) is the reciprocal of a degree $2N$ polynomial.

Example 4.6. Below we shall prove Corollary 4.11, confirming the assertions made in Example 0.3.

We begin with some rather trivial topological discussions. In (4.7) – (4.9), complex algebraic varieties are equipped with analytic topology, and singular cohomology groups are assumed to have coefficients in $\mathbf{Q}$.

The first preparation concerns about computing relative cohomology using a compactification. Let $f\colon \overline{X} \to \mathbf{A}^1$ be a proper, generically smooth morphism of algebraic varieties over $\mathbf{C}$. Let X be an open subvariety of $\overline{X}$, and $E = \overline{X} - X$. Suppose that there is a neighborhood T of E such that $f|_T$ and $f|_{T-E}$ are locally topologically trivial fibrations (hence are trivial as $\mathbf{A}^1$ is contractible). Let $\overline{F}$ be a generic fiber of f and $F = \overline{F} \cap X$.



Lemma 4.7. *Notation as above, the natural map* $\mathrm{H}^i(\overline{X}, \overline{F}) \to \mathrm{H}^i(X, F)$ *is an isomorphism for all* i.

Proof. This is a simple application of excision. To begin with, since $T \to \mathbf{A}^1$ is a trivial fibration, there is a deformation retract from T onto $T \cap \overline{F}$. This induces a homotopy equivalence between $T \cup \overline{F}$ and $\overline{F}$. Thus the pair $(\overline{X}, T \cup \overline{F})$ and the pair $(\overline{X}, \overline{F})$ have the same cohomology.

Since E is contained in the interior of $T \cup \overline{F}$, excision implies that the pair $(\overline{X}, T \cup \overline{F})$ and the pair $(X, (T - E) \cup F)$ have the same cohomology. Using the fact that $T - E \to \mathbf{A}^1$ is a trivial fibration, we find a deformation retract from $(T - E) \cup F$ onto F. Thus the pair $(X, (T - E) \cup F)$ and the pair (X, F) have the same cohomology. This completes the proof. □

Next, we explain how to construct $\overline{X}$ and $\overline{F}$ used in Lemma 4.7 from the standard situation.

Construction 4.8. Let P be a smooth proper scheme over a subfield of $\mathbf{C}$ of pure dimension d. Suppose the divisors $D_1, \ldots, D_r$ and X_0 define the standard situation. For any subset J of $\{1, 2, \ldots, r\}$, let D_J be the intersection of $\bigcap_{j \in J} D_j$. Then D_J is a closed subscheme of P who dimension equals $d - \operatorname{Card} J$.

The base locus of the pencil $\Pi_0 = \{uX_0 + v \sum e_i D_i\}_{[u,v] \in \mathbf{P}^1}$ is $Z = X_0 \cap (D_1 \cup \cdots \cup D_r)$. In general, blowing up the base locus will result a singular scheme. Instead, we resort to an iterated blowing up which is defined inductively.

To begin with, let $P_0 = P$.

Let $Z_0 = X_0 \cap \bigcup_{\operatorname{Card} J = d-1} D_J$. By hypothesis, Z_0 is a nonsingular closed subscheme of P_0. Let P_1 be the blowup of P_0 along Z_0, and $\varpi_1\colon P_1 \to P$ the projection morphism. Now we consider the pencil Π_1 defined by $\widetilde{X}_0 = \varpi_1^* X_0$ and $\varpi_1^* \sum e_i D_i = \sum e_i \widetilde{D}_i + e_{r+1}\widetilde{D}_{r+1}$, where $\widetilde{D}_{r+1}$ is the exceptional divisor, and $e_{r+1} = \sum_{i=1}^r e_i$.

The transversality hypothesis ensures that these divisors still define a standard situation on the scheme P_1. In this new situation, the intersection $\widetilde{X}_0 \cap \widetilde{D}^{(d-1)}$ is empty, and the intersection $\widetilde{X}_0 \cap \widetilde{D}_J$, where J is a subset of $\{1, 2, \ldots, r+1\}$, Card $J = d-2$, is a nonsingular curve. These curves are necessarily pairwise disjoint. Thus $Z_1 = \widetilde{X}_0 \cap \bigcup_{\mathrm{Card}\, J = d-2} \widetilde{D}_J$ is a disjoint union of nonsingular curves. Let P_2 be the blowup of P_1 along Z_1, with projection morphism $\varpi_2 \colon P_2 \to P_1$

On P_2, we continue to blow up disjoint surfaces like before. Continuing this procedure produces a sequence of blowup schemes

$$\varpi \colon P_{d-2} \xrightarrow{\varpi_{d-2}} P_{d-3} \to \cdots \to P_1 \xrightarrow{\varpi_1} P_0.$$

As each blowup has a smooth center, P_{d-2} is a smooth proper scheme. As the inverse image of the intersection $X_0 \cap \bigcup_{i=1}^r D_i$ in P_{d-2} is a Cartier divisor, ϖ factors through the blowup $\mathrm{Bl}_Z P_0$, where Z is the base locus of the original pencil Π_0. Thus the morphism $f^{\mathrm{pre}} \colon \mathrm{Bl}_Z P_0 \to \mathbf{P}^1$ coming from the pencil Π_0 gives rise to a morphism $f \colon P_{d-2} \to \mathbf{P}^1$. Note that for $u \neq 0$, $f^{\mathrm{pre}-1}\{[u,v]\} = f^{-1}\{[u,v]\}$.

Let $\overline{X} = P_{d-2} - f^{-1}(\infty)$ ($\infty = [0,1] \in \mathbf{P}^1$). There is a generically smooth morphism $f \colon \overline{X} \to \mathbf{A}^1$ and a projection $\pi \colon \overline{X} \to P$.

Now we are in the situation considered in Lemma 4.7. Here E is the intersection of the exceptional divisor of $P_{d-2} \to P_0$ with $\overline{X}$. By construction, it has a "tubular neighborhood" T (e.g., the preimage of a tubular neighborhood of $X_0 \cap (\bigcup D_i)$) within $\overline{X}$ such that the restriction of f to T is a topologically trivial fibration.

Retain the above notation. Let $s_\infty = \prod s_i^{e_i}$. Let $X = P - \bigcup D_i$. Then via the morphism π, we can regard X as an open subscheme of $\overline{X}$. The restriction of f to X is given by the ratio $g = s_0/s_\infty$. Then by Lemma 4.7, for t generic we have

$$\mathrm{H}^i(\overline{X}, X_t) \cong \mathrm{H}^i(X, g^{-1}(t)) \cong \mathrm{H}^i(\overline{X}, g^{-1}(0)).$$

In practice, we are more interested in considering the function g on X; and the construction above allows us to construct a proper function which is ready for taking reduction modulo p.

The next lemma tells us that in a certain preferable situation, the calculation of cohomology groups reduces to the calculation of the Euler characteristics.

Lemma 4.9. *Notation as in Construction 4.8, assume in addition that X and $P - X_0$ are affine (e.g., when the invertible sheaf $\bigotimes_{i=1}^r \mathcal{L}_i^{\otimes e_i}$ is ample). Then $\mathrm{H}^i(X, g^{-1}(0))$ is nonzero only if $i = \dim X$.*

The Euler characteristic of $\mathrm{H}^\bullet(X, g^{-1}(0))$ is

$$(-1)^{\dim P} \int_P \frac{c((\Omega_P^1)^\vee)}{(1 + \sum_{i=1}^r e_i c_1(\mathcal{L}_i)) \prod_{i=1}^r (1 + c_1(\mathcal{L}_i))}.$$

See for example [21, Theorem 5.4.1]. If the hypothesis of Lemma 4.9 is fulfilled, then the absolute value of this number is also the dimension of $\mathrm{H}^{\dim X}(X, g^{-1}(0))$.

Proof of Lemma 4.9. Since X and $g^{-1}(0)$ are smooth affine varieties, the relative cohomology $\mathrm{H}^i(X, g^{-1}(0))$ vanishes if $i > \dim X$. It suffices to prove the relative cohomology also vanishes when $i < \dim X$.

For any subset J of $\{1,2,\dots,r\}$, let $D_J = \bigcap_{j\in J} D_j$. Let $D^{(p)} = \coprod_{\mathrm{Card}\, J=p} D_J$, and write $D^{(0)} = P$. The scheme $D^{(p)}$ is smooth proper of dimension $n-p$, and the natural morphism $D^{(p)} \to P$ is affine.

There exists a spectral sequence

$$E_1^{-p,q} = \mathrm{H}^{q-2p}(D^{(p)}, D^{(p)} \times_P X_0) \Rightarrow \mathrm{H}^{q-p}(X, g^{-1}(0)). \tag{4.10}$$

Granting the existence of this spectral sequence, let us finish the proof. Since $P - X_0$ is affine, $D^{(p)} - D^{(p)} \times_P X_0$ are affine for all p. If $i < \dim X$, then $i-p < \dim D^{(p)}$. By Artin's vanishing theorem, we have

$$\mathrm{H}^{i-p}(D^{(p)}, D^{(p)} \times_P X_0) = \mathrm{H}_c^{i-p}(D^{(p)} - D^{(p)} \times_P X_0) = \{0\}.$$

It follows that $E_1^{-p,q} = 0$ if $q-p < \dim X$. This implies that $\mathrm{H}^i(X, g^{-1}(0))$ vanishes when $i < \dim X$, as desired. The construction of the spectral sequence (4.10) will be recalled in Paragraph 4.12 at the end of this section. □

Having discussed some topology, we now return to Example 0.3. We henceforth enforce the notation set up there.

Corollary 4.11. *Assume $\mathfrak{p}$ is a prime of $\mathbf{K}$ satisfying the two conditions (0.3/1) and (0.3/2). Let p be the residue characteristic of $\mathfrak{p}$, and $\mathbf{K}_\mathfrak{p}$ be the completion of $\mathbf{K}$ at $\mathfrak{p}$.*

(1) *The dimension of the rigid cohomology $\mathrm{H}^i_{\mathrm{rig}}(X \otimes \mathbf{K}_\mathfrak{p}(\zeta_p)/\mathbf{K}_\mathfrak{p}(\zeta_p), \mathcal{L}_\pi)$ equals the dimension of $\mathrm{H}^i(X^{\mathrm{an}}_{\mathbf{C}}, g^{-1}(0)^{\mathrm{an}}_{\mathbf{C}})$.*

(2) *If both X and $P - X_0$ are affine, the rigid cohomology is nonzero only in cohomology degree $\dim X$, and its dimension over $\mathbf{K}_\mathfrak{p}$ is*

$$\int_P \frac{c((\Omega^1_P)^\vee)}{(1+\sum_{i=1}^r e_i c_1(\mathcal{L}_i))\prod_{i=1}^r (1+c_1(\mathcal{L}_i))}.$$

Thus, the L-series associated with the function g modulo $\mathfrak{p}$ is a polynomial or a reciprocal of a polynomial, whose degree equals $\int_P \frac{c((\Omega^1_P)^\vee)}{(1+\sum_{i=1}^r e_i c_1(\mathcal{L}_i))\prod_{i=1}^r (1+c_1(\mathcal{L}_i))}$.

If $\mathcal{L}_i$ are ample, the assertion (2) is due to N. Katz [21] (without the liftablity hypothesis).

Proof of Corollary 4.11. We could assume that $\mathbf{K}$ contains a p^{th} root of unity. Perform Construction 4.8. The morphism

$$f \colon P_{d-2} \otimes \mathcal{O}_{\mathbf{K}_\mathfrak{p}} \to \mathbf{P}^1_{\mathcal{O}_{\mathbf{K}_\mathfrak{p}}}$$

then satisfies the hypothesis of Theorem 3.15. Thus the theorem implies the rigid cohomology associated with the reduction of g has the same dimension as the twisted algebraic de Rham cohomology over $\mathbf{K}_\mathfrak{p}$.

By performing a base extension to $\mathbf{C}$, using the isomorphism provided by (0.1/1) and (0.1/2), we know that the dimension of the twisted algebraic de Rham cohomology defined by g is equal to the dimension of the relative cohomology $\mathrm{H}^i(X^{\mathrm{an}}_{\mathbf{C}}, g^{-1}(0)^{\mathrm{an}}_{\mathbf{C}})$. Here using $t=0$ instead of a generic t is legal, because by construction 0 is a typical value of g; see the footnote on page 1.

The second assertion follows from Lemma 4.9 and the trace formula. □

A particular instance of the standard situation is as follows. Let $P = \mathbf{P}^n$, X_∞ be the union of coordinate lines defined by $z_0 z_1 \cdots z_n = 0$, and let X_0 be the Fermat hypersurface of degree $n+1$. Then the function

$$z \mapsto \frac{z_0^{n+1} + z_1^{n+1} + \cdots + z_n^{n+1}}{z_0 \cdots z_n} : \mathbf{G}_{\mathrm{m}}^n \to \mathbf{A}^1$$

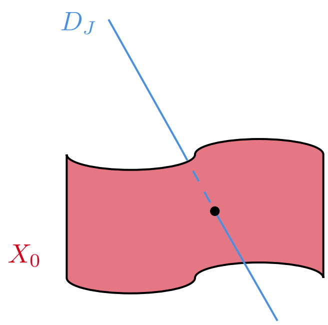
fits the standard situation. Assuming the residue characteristic of $\mathfrak{p}$ is not a factor of $n+1$, then we can apply Theorem 3.15. Since this is the "ample case", Lemma 4.9 implies that the rigid cohomology is trivial except in degree n, and its dimension equals $(-1)^n$ times the Euler characteristics, which is $n^n(n+1)$. Of course, this computation may already be deduced from Katz's theorem.

4.12. We recall a construction of the spectral sequence (4.10) for convenience of the reader. Notation and conventions will be as in Lemma 4.9.

Consider the natural simplicial morphism $a \colon D^{(\bullet+1)} \to P$, viewing P as a constant simplicial scheme. Then the relative cohomology of the pair $(X, g^{-1}(0))$ is computed by the homotopy cofiber of the morphism

$$a_* a^! v_! \mathbf{Q} \to v_! \mathbf{Q},$$

where $v \colon P - X_0 \to P$ is the open immersion.



The complex

$$a_* a^! v_! \mathbf{Q} = [\cdots \to a_{2*} a_2^! v_! \mathbf{Q} \to a_{1*} a_1^! v_1 \mathbf{Q}]$$

can be understood explicitly using the fact that D_J intersects X_0 transversely. Indeed, if a_J is the inclusion morphism from D_J into P, then $a_i = \coprod_{\mathrm{Card}\, J = i} a_J$, and (we use $\vee$ for Verdier dual)

$$a_{J*} a_J^! v_! \mathbf{Q}[\dim X] = a_{J*}(a_J^* v_* \mathbf{Q}[\dim X])^\vee.$$

Let $v_J \colon D_J - D_J \cap X_0 \to D_J$ be the inclusion morphism.

We claim that $a_J^* R v_* \mathbf{Q} \cong R v_{J*} \mathbf{Q}$. Indeed, consider the following fiber diagram,

$$\begin{array}{ccc} X_0 \cap D_J & \longrightarrow & X_0 \\ \big\downarrow{\scriptstyle \iota_J} & & \big\downarrow{\scriptstyle \iota} \\ D_J & \xrightarrow{\ a_J\ } & P \end{array},$$

and the distinguished triangle

$$\iota_* \iota^! \mathbf{Q} \to \mathbf{Q} \to R v_* \mathbf{Q} \to .$$

Since X_0 is a smooth divisor in P, the Thom isomorphism theorem implies that $\iota^!\mathbf{Q}_P = \mathbf{Q}_{X_0}[-2]$. Pulling back the above distinguished triangle by a_J yields a distinguished triangle

$$a_J^*\iota_*\mathbf{Q}_{X_0}[-2] \to \mathbf{Q}_{D_J} \to a_J^*Rv_*\mathbf{Q} \to .$$

Since ι is proper, and since $X_0 \cap D_J$ is smooth of codimension 1, we have

$$\begin{aligned} a_J^*\iota_*\mathbf{Q}_{X_0}[-2] &= \iota_{J*}\mathbf{Q}_{X_0\cap D_J}[-2] \\ &= \iota_{J*}\iota_J^!\mathbf{Q}_{D_J}. \end{aligned}$$

By adjunction, (Hom being taken in the derived category) $\mathrm{Hom}(\iota_{J*}\iota_J^!\mathbf{Q}_{X_0}, \mathbf{Q}_{X_0})$ equals $\mathrm{Hom}(\iota_J^!\mathbf{Q}, \iota_J^!\mathbf{Q}) = \mathrm{H}^0(X_0 \cap D_J, \mathbf{Q})$. Thus up to a nonzero constant multiple on each connected component of $X_0 \cap D_J$, there exists only one nonzero morphism from $\iota_{J*}\iota_J^!\mathbf{Q}$ to $\mathbf{Q}$ in the derived category. (In fact, using the existence of integral structure, the term "up to a nonzero constant" could be replaced by "up to sign".) Therefore the homotopy cofiber of the morphism

$$\begin{array}{ccc} a_J^*\iota_*\mathbf{Q}_{X_0}[-2] & \longrightarrow & \mathbf{Q}_{D_J} \\ \| & & \\ \iota_{J*}\iota_J^!\mathbf{Q}_{D_J} & & \end{array}$$

is necessarily isomorphic to the homotopy cofiber of the canonical morphism

$$\iota_{J*}\iota_J^!\mathbf{Q}_{D_J} \to \mathbf{Q}_{D_J}$$

which is $Rv_{J*}\mathbf{Q}$. This justifies the claim.

Thus, letting $p = \mathrm{Card}\, J$ be the codimension of D_J, we have

$$a_{J*}a_J^!v_!\mathbf{Q} = a_{J*}(v_{J!}\mathbf{Q})[-2p].$$

The complex $a_{J*}(v_{J!}\mathbf{Q})$ is precisely the derived incarnation of the relative cohomology $\mathrm{H}^\bullet(D_J, D_J \cap X_0)$. The spectral sequence (associated with the "filtration bête") of the complex $a_*a^!v_!\mathbf{Q} \to \mathbf{Q}$ gives the spectral sequence (4.10).

5. A remark on Higgs cohomology

Let $f\colon X \to \mathbf{A}^1$ be a nonconstant morphism from a smooth algebraic variety of pure dimension n over a field k to the affine line. We have been studying cohomological objects that are related to the irregular connection ∇_f. In the theory of twisted algebraic de Rham cohomology, one can associate an irregular Higgs field to the connection ∇_f, i.e., the Higgs field defined by $\wedge \mathrm{d}f$. Thus, in conjunction with the de Rham complex of ∇_f one may also study the algebraic Higgs complex

$$\Omega_X^0 \xrightarrow{\wedge \mathrm{d}f} \Omega_X^1 \xrightarrow{\wedge \mathrm{d}f} \cdots \xrightarrow{\wedge \mathrm{d}f} \Omega_X^n.$$

A celebrated theorem of Barannikov–Kontsevich (the first published proof is due to Sabbah [31]; for a p-adic proof, see Ogus–Vologodsky [28]) asserts that the algebraic Higgs cohomology and the twisted algebraic de Rham cohomology have the same dimension.

In this section we explain how to transplant this to the rigid analytic world. We shall compare a "dagger" variant of the Higgs cohomology with the algebraic Higgs cohomology. Then in the nice situation, Theorem 0.2 allows us to relate twisted rigid cohomology and the dagger Higgs cohomology.

In the work of Adolphson–Sperber [1] on exponential sums, one also finds the use of algebraic Higgs cohomology. In fact their finiteness theorem is deduced from the finiteness of the Higgs cohomology of the reduction. In contrast, the result of this section happens completely on the generic fiber, thereby does not really concern whether the pole divisor has good reduction or not.

5.1. Notation.

- Let K be a discrete valuation field of characteristic 0 whose residue characteristic is p. Let $\mathfrak{X}$ be a scheme smooth over a discrete valuation ring $\mathcal{O}_K$.
- Let $f\colon \mathfrak{X} \to \mathbf{A}^1_R$ be a *proper* function admitting a compactification $\overline{f}\colon \overline{\mathfrak{X}} \to \mathbf{P}^1_{\mathcal{O}_K}$, in which $\overline{\mathfrak{X}}$ is smooth over R.
- Also denote by $\overline{f}\colon \overline{X} \to \mathbf{P}^1_K$ and $f\colon X \to \mathbf{P}^1_K$ be the restriction of f to the generic fibers $\overline{X}$ and X of $\overline{\mathfrak{X}}$ and $\mathfrak{X}$ respectively.
- Let V_r be the inverse image $f^{-1}(\mathbf{D}^+(0;r))$ of the rigid analytic disk under f. Thus V_r is an rigid analytic subspace of $\overline{X}$.

5.2. Hypothesis. We assume that $f\colon X \to \mathbf{A}^1_K$ has no critical values in $\mathbf{D}^-(\infty;1)$.

Note that we do not enforce the hypothesis that the components of the pole divisor have good reduction anymore.

5.3. We can consider three types of Higgs cohomology.

(1) The algebraic Higgs cohomology
$$R\Gamma(X, (\Omega^\bullet_X, \mathrm{d}f))$$
of X.

(2) A dagger version of the Higgs cohomology
$$R\Gamma(\overline{X}^{\mathrm{an}}, j_1^\dagger(\Omega^\bullet_{\overline{X}^{\mathrm{an}}}, \mathrm{d}f)),$$
where for a real number r, $j_r\colon V_r \to \overline{X}$ denotes the open inclusion.

(3) The analytic Higgs cohomology
$$R\Gamma(V_r, (\Omega^\bullet_{V_r}, \mathrm{d}f)) \quad (r > 1).$$

Proposition 5.4. *Notation as above, the natural morphisms*
$$R\Gamma(X, (\Omega^\bullet_X, \mathrm{d}f)) \to R\Gamma(V_r, (\Omega^\bullet_{V_r}, \mathrm{d}f)) \to R\Gamma(\overline{X}^{\mathrm{an}}, j_1^\dagger(\Omega^\bullet_{\overline{X}^{\mathrm{an}}}, \mathrm{d}f))$$
are quasi-isomorphisms.

Proof. It suffices to prove the first arrow is a quasi-isomorphism, as the third item is obtained from the second by taking colimit with respect to $r \to 1^-$.

Let P be the pole divisor of $f\colon \overline{X} \to \mathbf{P}^1_K$. For any connected subset I of $\mathbf{R}_{\geq 0}$, let T_I be the inverse image of the rigid analytic annulus $\Delta_I(\infty)$ centered at $\infty \in \mathbf{P}^1$. Thus
$$T_I = f^{-1}(\Delta_I(\infty)).$$
Let $\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(*P)$ be the subcomplex of $j_*\Omega^\bullet_{X^{\mathrm{an}}}$ consisting of differential forms with at worst poles along P. Then by rigid analytic GAGA, the natural morphism of complexes
$$R\Gamma(X, (\Omega_X, \mathrm{d}f)) \to \underbrace{\Gamma(\overline{X}^{\mathrm{an}}, (\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(*P), \mathrm{d}f))}_{\text{“moderate Higgs complex”}}$$

is a quasi-isomorphism.

Choose a function $r \mapsto \delta_r :]1,\infty[\to \mathbf{R}$ such that $1 > \delta_r > r^{-1}$. For each $r > 1$, V_r and $T_{[0,\delta_r]}$ form an admissible cover of $\overline{X}$. By Mayer–Vietoris, the complex $R\Gamma(\overline{X}^{\mathrm{an}}, (\Omega^\bullet_{\overline{X}^{\mathrm{an}}}(*P), \mathrm{d}f))$ is the homotopy kernel of

$$R\Gamma(V_r, (\Omega^\bullet_{V_r}, \mathrm{d}f)) \oplus R\Gamma(T_{[0,\delta_r]}, (\Omega^\bullet_{T_{[0,\delta_r]}}(*P), \mathrm{d}f)) \to R\Gamma(T_{[r^{-1},\delta_r]}, (\Omega^\bullet_{T_{[r^{-1},\delta_r]}}, \mathrm{d}f)))$$

We shall show that the natural morphism

$$R\Gamma(T_{[0,\delta_r]}, (\Omega^\bullet_{T_{[0,\delta_r]}}(*P), \mathrm{d}f)) \to R\Gamma(T_{[r^{-1},\delta_r]}, (\Omega^\bullet_{T_{[r^{-1},\delta_r]}}, \mathrm{d}f)))$$

is a quasi-isomorphism (in fact, we shall show both are acyclic).

Below we shall write $a = r^{-1}$, $b = \delta_r$. Thus $0 < a < b < 1$. Let $U = \mathrm{Sp}(A)$ be an affinoid subdomain of $T_{[0,b]}$ admitting an étale morphism to the disk $\mathbf{D}^+(0;1)^n$. Then $W = U \cap T_{[a,b]}$ is an affinoid subdomain of U as $T_{[a,b]} \to T_{[0,b]}$ is an affinoid morphism. It suffices to prove the restrictions of the two Higgs complexes on respectively U and W are acyclic.

Write $W = \mathrm{Sp}(B)$. Then the morphism $f : \overline{X} \to \mathbf{P}^1$ gives rise to an element in A, and an element in B via the morphism $A \to B$. With respect to the the coordinate system provided by the étale morphism $U \to \mathbf{D}^+(0;1)^n$, the Higgs complex of B is the Koszul complex $\mathrm{Kos}(B^{\oplus n}; \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n})$ associated with the partial derivatives $\partial f/\partial x_i$. Since f is smooth, the Jacobian ideal (i.e., the ideal formed by the partials derivatives of f) cuts out the empty rigid analytic space. By Nullstellensatz for affinoid algebras, the Jacobian ideal equals the unit ideal of B. The acyclicity of the Koszul complex then follows from a standard algebra lemma, Lemma 5.5 below.

The exactness of the "moderate Higgs complex" on $\mathrm{Sp}(A)$ is acyclic is proved similarly. We apply Lemma 5.5 to $A[1/f]$ and the partial derivatives of f. Again, since f is smooth, the ordinary Hilbert Nullstellensatz implies that the Jacobian ideal of f form the unit ideal of $A[1/f]$. □

The following is well-known.

Lemma 5.5. *Let R be a noetherian commutative ring. Let $a_1, \dots, a_n \in R$ be elements such that $\sum a_i R = R$. Then the Koszul complex* $\mathrm{Kos}(R; a_1, \dots, a_n)$ *is exact in every degree.*

Proof. We can assume R is local since we can check the exactness at every prime. In this local situation, our condition is equivalent to saying that at least one of the elements a_i is a unit in R, hence without loss of generality we may assume a_1 is a unit in R. Since we have a factorization:

$$\mathrm{Kos}(R; a_1, \dots, a_n) = \bigotimes_R [R \xrightarrow{\cdot a_i} R],$$

it suffices to note that $R \xrightarrow{\cdot a_1} R$ is an acyclic complex. □

We leave the apparent generalization to non-proper functions (using logarithmic forms) to the reader. It should be mentioned that one cannot use algebraic forms in the non-proper case to calculate the Higgs cohomology, as non-isolated critical points will contribute infinite dimensional cohomology.

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